

# MEASURES OF LOCALIZATION AND QUANTITATIVE NYQUIST DENSITIES

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**ABSTRACT.** We obtain a refinement of the degrees of freedom estimate of Landau and Pollak. More precisely, we estimate, in terms of  $\epsilon$ , the increase in the degrees of freedom resulting upon allowing the functions to contain a certain prescribed amount of energy  $\epsilon$  outside a region delimited by a set  $T$  in time and a set  $\Omega$  in frequency. In this situation, the lower asymptotic Nyquist density  $|T| |\Omega| / 2\pi$  is increased to  $(1 + \epsilon) |T| |\Omega| / 2\pi$ . At the technical level, we prove a *pseudospectra* version of the classical spectral dimension result of Landau and Pollak, in the multivariate setting of Landau. Analogous results are obtained for Gabor localization operators in a compact region of the time-frequency plane.

*“It is easy to argue that real signals must be bandlimited. It is also easy to argue that they cannot be so”, David Slepian, On Bandwidth, 1976.*

## 1. INTRODUCTION

**1.1. The Nyquist rate and Landau-Pollack degrees of freedom estimate.** Let  $D_T$  and  $B_\Omega$  denote the operators which cut the time content outside  $T$  and the frequency content outside  $\Omega$ , respectively. In the fundamental paper [15], whose purpose was to *examine the true in the engineering intuition that there are approximately  $|T| |\Omega| / 2\pi$  independent signals of bandwidth  $\Omega$  concentrated on an interval of length  $T$* , Landau and Pollak have considered the eigenvalue problem associated with the positive self-adjoint operator

$$(1.1) \quad P_{T,\Omega} = D_T B_\Omega D_T$$

When  $T$  and  $\Omega$  are real intervals, the operator involved in this problem can be written explicitly as

$$(P_{T,\Omega} f)(x) = \begin{cases} \int_T \frac{\sin \Omega(x-t)}{\pi(x-t)} f(t) dt & \text{if } x \in T \\ 0 & \text{if } x \notin T \end{cases}.$$

The cornerstone of the results in [15] is the following asymptotic estimate for the number of eigenvalues  $\lambda_n$  of (1.1) which are close to one:

$$(1.2) \quad \#\{n : \lambda_n > 1 - \delta\} \simeq |T| |\Omega| / 2\pi + C_\delta \log(|T| |\Omega|),$$

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as  $T \rightarrow \infty$ , where  $C_\delta$  is a constant depending only on  $\delta$ . Since the eigenvalues of the operator (1.1) are the same as those of the operator  $B_\Omega D_T$ , whose eigenfunctions  $f$  satisfy

$$\int_T |f|^2 = \lambda \|f\|^2,$$

the estimate (1.2) provides us with the number of orthogonal eigenfunctions  $f$  of (1.1), such that

$$\int_T |f|^2 \sim \|f\|^2,$$

asymptotically when  $T \rightarrow \infty$ . Within mathematical signal analysis (see, for instance the discussion in [5, pg. 23] and the recent book [12]), (1.2) is viewed as a mathematical formulation of the Nyquist rate, the fact that a time- and bandlimited region  $T \times \Omega$  corresponds to  $|T| |\Omega| / 2\pi$  degrees of freedom. In other words, there exist, up to a small error,  $|T| |\Omega| / 2\pi$  independent functions that are essentially timelimited to  $T$  and bandlimited to  $\Omega$ .

The main goal of this note is to refine the degrees of freedom estimate (1.2) in the context to be made precise in the next subsection.

**1.2. A refinement of Landau-Pollack degrees of freedom estimate.** Ideally, one would like to count the number of orthogonal functions in  $L^2(\mathbb{R})$ , which are time and band-limited to a bounded region like  $T \times \Omega$ . Unfortunately, such functions do not exist (because band-limited functions are analytic). As a result, it is natural to count the number of orthonormal functions in  $L^2(\mathbb{R})$ , which are approximately time and band-limited to a bounded region like  $T \times \Omega$ . An optimal solution to this problem is given by the number of eigenfunctions of (1.1) whose eigenvalues are very close to one in the sense that they exceed a threshold  $1 - \delta$ , leading to (1.2). We remark that estimate (1.2) counts the degrees of freedom in spaces generated by the so-called prolate spheroidal wave functions (see [16] for a recent reference on these functions). Our count of degrees of freedom will be based on different functions (but the new functions will be constructed using the prolate spheroidal functions).

Our purpose is to refine (1.2), by taking advantage of the fact that most of the eigenvalues of  $P_{T,\Omega}$  are closer to 1 than to  $1 - \delta$ . To get an estimate of the space of functions satisfying simply  $\|P_{T,\Omega} f - f\| \geq 1 - \delta$ , we can replace  $n$  orthogonal eigenfunctions (which, in the case of the interval described in this introduction are the prolate spheroidal wave functions [16]) of  $P_{T,\Omega}$  whose eigenfunctions are close to 1, with  $n + 1$  orthogonal functions with  $\|P_{T,\Omega} f - f\| \approx 1 - \delta$ . Essentially, we split the well concentrated energy of the  $n$  prolate functions among  $n + 1$  vectors and add an extra dimension to obtain an orthogonal set. This idea will allow one to build a set of orthonormal functions in  $L^2(\mathbb{R})$ , which is a bit less concentrated than the prolates, so that it contains a prescribed *quantity  $\epsilon$  of time-frequency content outside the bounded region  $T \times \Omega$* . Precisely, we will count the number of orthogonal functions in  $L^2(\mathbb{R})$ ,

$\epsilon$ -localized in the sense that

$$(1.3) \quad \|P_{T,\Omega}f - f\|^2 \leq \epsilon.$$

From our main result it follows that (1.2) has the following analogue in this setting: if  $\eta(\epsilon, T, \Omega)$  stands for the maximum number of orthogonal functions of  $L^2(\mathbb{R})$  satisfying (1.3), then, as  $|T| \rightarrow \infty$ ,

$$(1.4) \quad \frac{|T||\Omega|}{2\pi}(1 + \epsilon) + C_\delta \log(|T||\Omega|) \leq \eta(\epsilon, T, \Omega) \leq \frac{|T||\Omega|}{2\pi}(1 - 2\epsilon)^{-1} + C_\delta \log(|T||\Omega|).$$

**1.3. Localization operators.** Our understanding of the concentration problem is based on the study of operators which localize signals in bounded regions of the time-frequency plane. Such operators are known in a broad sense as time-frequency localization operators; their eigenfunctions are orthogonal sequences of functions with optimal concentration properties. The quantitative formulation of the concentration problem can be seen in terms of localization operators as follows: rather than looking for the optimal concentrated functions in a given region of the time-frequency plane, we will allow the functions to contain a certain prescribed amount of energy outside the given region, and estimate the resulting increase in the degrees of freedom. Given an operator  $L$ , instead of counting the eigenfunctions of

$$Lf = \lambda f$$

associated with eigenvalues  $\lambda$  close to one, we will count orthogonal functions  $\epsilon$ -localized with respect to  $L$  in the sense that

$$(1.5) \quad \|Lf - f\|^2 \leq \epsilon.$$

In the next paragraph we will see how the idea of  $\epsilon$ -localization arises from the concept of pseudospectra of linear operators.

**1.4. Pseudospectra and  $\epsilon$ -localization.** The result of Landau and Pollak has later been improved by Landau to several dimensions and more general sets than intervals in [13] and [14]. Also in [14], Landau introduced the concept of  $\epsilon$ -approximated eigenvalues and eigenfunctions. This concept is a forerunner of what is nowadays known as the *pseudospectra* in the numerical analysis of non-normal matrices [20]. Recent developments in spectral approximation theory involve the concept of  $n$ -pseudospectrum, which has been introduced in [11] with the purpose of approximating the spectrum of bounded linear operators on an infinite dimensional, separable Hilbert space, and then used in the proof of the computability of the spectrum of a linear operator on a separable Hilbert space [10]. We will recall Landau's original definition, which was the following:

**Definition 1.**  $\lambda$  is an  $\epsilon$ -approximated eigenvalue of  $L$  if there exists  $f$  with  $\|f\| = 1$ , such that  $\|Lf - \lambda f\| \leq \epsilon$ . We call  $f$  an  $\epsilon$ -approximated eigenfunction corresponding to  $\lambda$ .

Thus, our quantitative measure (1.5) for the time-frequency localization of  $f$  is equivalent to  $f$  being a  $\epsilon$ -approximated eigenfunction corresponding to 1.

**Example 1.** *Suppose that  $\varphi$  is an eigenfunction of  $P_{rT,\Omega}$  with eigenvalue  $\lambda$ . Then*

$$\|P_{rT,\Omega}\varphi - \varphi\| = 1 - \lambda.$$

*Thus, every eigenfunction of  $P_{rT,\Omega}$  is a  $(1-\lambda)$ -pseudoeigenfunction of  $P_{rT,\Omega}$  with pseudoeigenvalue 1.*

The relevant fact is that the number of orthogonal pseudoeigenfunctions with pseudoeigenvalue greater than a given threshold is larger than the number of eigenfunctions with eigenvalue greater than that threshold. A large class of functions satisfying (1.2) arises from the set of almost bandlimited functions in the sense of Donoho-Stark's concept of  $\epsilon$ -concentration.

**Example 2.** *According to [7],  $f$  is  $\epsilon_T$ -concentrated in  $T$  if*

$$\|D_T f - f\| \leq \epsilon_T$$

*and its Fourier transform  $Ff$  (see definitions in the next section) is  $\epsilon_\Omega$ -concentrated in  $\Omega$  if*

$$(1.6) \quad \|B_\Omega f - f\| \leq \epsilon_\Omega.$$

*An application of the triangle inequality shows that if  $f$  is  $\epsilon_T$ -concentrated in  $T$  and  $Ff$  is  $\epsilon_\Omega$ -concentrated in  $\Omega$  then*

$$\|B_\Omega D_T f - f\| \leq \epsilon_T + \epsilon_\Omega.$$

*and another application of the triangle inequality gives*

$$(1.7) \quad \|P_{rT,\Omega} f - f\| \leq 2\epsilon_T + \epsilon_\Omega.$$

*Thus, if  $f$  is  $\epsilon_T$ -concentrated in  $T$  and  $Ff$  is  $\epsilon$ -concentrated in  $\Omega$ , then  $f$  is a  $(2\epsilon_T + \epsilon_\Omega)$ -pseudoeigenfunction of  $P_{rT,\Omega}$  with pseudoeigenvalue 1.*

One should notice that these notions, as well as the topic investigated in this note, can be related to Slepian's philosophical and mathematical quest [19], aiming at solving the bandwidth paradox: “*It is easy to argue that real signals must be bandlimited. It is also easy to argue that they cannot be so*” [19]

**1.5. Organization of the paper.** This is essentially a single-result paper, which is Theorem 1 in the next section. We first provide some background concerning Landau's results about the extension of the time-band limiting problem to functions in  $\mathbb{R}^d$ , bandlimited to a set of finite measure and the main notations. The last section of the paper is devoted to another important class of operators where our results apply, namely Gabor localization operators. Since the proofs for Gabor localization operators are very similar to those in section 2, they are omitted.

## 2. NOTATIONS AND MAIN RESULTS

**2.1. Time- and band- limiting operators.** A description of the general set-up of [13] and [14] follows. The sets  $T$  and  $\Omega$  are general subsets of finite measure of  $\mathbb{R}^d$ . Let

$$Ff(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(t) e^{-i\xi t} dt$$

denote the Fourier transform of a function  $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ . The subspaces of  $L^2(\mathbb{R}^d)$  consisting, respectively, of the functions supported in  $T$  and of those whose Fourier transform is supported in  $\Omega$  are

$$\begin{aligned} \mathcal{D}(T) &= \{f \in L^2(\mathbb{R}^d) : f(x) = 0, x \notin T\} \\ \mathcal{B}(\Omega) &= \{f \in L^2(\mathbb{R}^d) : Ff(\xi) = 0, \xi \notin \Omega\}. \end{aligned}$$

Let  $D_T$  be the orthogonal projection of  $L^2(\mathbb{R}^d)$  onto  $\mathcal{D}(T)$ , given explicitly by the multiplication of a characteristic function of the set  $T$  by  $f$ :

$$D_T f(t) = \chi_T(t) f(t)$$

and let  $B_\Omega$  be the orthogonal projection of  $L^2(\mathbb{R}^d)$  onto  $\mathcal{B}(\Omega)$ , given explicitly as

$$B_\Omega f = F^{-1} \chi_\Omega F f = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} h(x-y) f(y) dy,$$

where  $Fh = \chi_\Omega$ . The following Theorem, comprising Lemma 1 and Theorem 1 of [14] gives important information concerning the spectral problem associated to the operator  $D_{rT} B_\Omega D_{rT}$ . This information will be essential in our proofs. The notation  $o(r^d)$  refers to behavior as  $r \rightarrow \infty$ .

**Theorem A** [14]. *The operator  $D_{rT} B_\Omega D_{rT}$  is bounded by 1, self-adjoint, positive, and completely continuous. Denoting its set of eigenvalues, arranged in nonincreasing order, by  $\{\lambda_k(r, T, \Omega)\}$ , we have*

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda_k(r, T, \Omega) &= r^d (2\pi)^{-d} |T| |\Omega| \\ \sum_{k=0}^{\infty} \lambda_k^2(r, T, \Omega) &= r^d (2\pi)^{-d} |T| |\Omega| - o(r^d). \end{aligned}$$

Moreover, given  $0 < \gamma < 1$ , the number  $M_r(\gamma)$  of eigenvalues which are not smaller than  $\gamma$ , satisfies, as  $r \rightarrow \infty$ ,

$$M_r(\gamma) = (2\pi)^{-d} |T| |\Omega| r^d + o(r^d).$$

We are now in a position to state and prove our main theorem. The lower inequality is proved by constructing a set of orthonormal functions of  $L^2(\mathbb{R}^d)$  satisfying (??). The

proof of the upper inequality uses some of the techniques contained in Landau's proof of the non-hermitian Szegő-type theorem [14, Theorem 3].

**Theorem 1.** *Let  $\eta(\epsilon, rT, \Omega)$  stand for the maximum number of orthonormal functions  $f \in L^2(\mathbb{R}^d)$  such that*

$$(2.1) \quad \|P_{rT, \Omega} f - f\|^2 \leq \epsilon.$$

*Then, as  $r \rightarrow \infty$ , the following inequalities hold:*

$$(2.2) \quad \frac{|T| |\Omega|}{(2\pi)^d} (1 + \epsilon) \leq \lim_{r \rightarrow \infty} \frac{\eta(\epsilon, rT, \Omega)}{r^d} \leq \frac{|T| |\Omega|}{(2\pi)^d} (1 - 2\epsilon)^{-1}.$$

*Proof.* We first prove the lower inequality in (2.2). Suppose (2.1) holds for a positive real  $\epsilon$ . Let  $\sigma > 0$  be such that  $\sigma^2 \leq \epsilon$  and let  $\mathcal{F} = \{\phi_k\}$  be the normalized system of eigenfunctions of the operator  $P_{rT, \Omega}$  with eigenvalues  $\lambda_k > 1 - \sigma$ . Now, given  $f \in L^2(\mathbb{R}^d)$ , write

$$(2.3) \quad f = \sum a_k \phi_k + h,$$

with  $h \in \text{Ker}(P_{rT, \Omega})$ . Then

$$(2.4) \quad P_{rT, \Omega} f = \sum a_k \lambda_k \phi_k$$

and

$$(2.5) \quad \begin{aligned} \|P_{rT, \Omega} f - f\|^2 &= \left\| \sum (1 - \lambda_k) a_k \phi_k + h \right\|^2 \\ &\leq \sigma^2 \sum |a_k|^2 + \|h\|^2 \\ &= \sigma^2 \|f\|^2 + (1 - \sigma^2) \|h\|^2. \end{aligned}$$

For the given  $\sigma > 0$  we pick a real number  $\gamma$  such that

$$(2.6) \quad \sigma^2 + (1 - \sigma^2)\gamma = \epsilon,$$

Writing (2.6) as

$$(2.7) \quad \gamma = \frac{\epsilon - \sigma^2}{1 - \sigma^2},$$

it is clear that  $\gamma$  is a positive increasing function of  $\sigma$ , and that  $\gamma \rightarrow \epsilon$  as  $\sigma \rightarrow 0$ . Now choose an integer number  $n$  such that

$$(2.8) \quad n \leq \frac{1}{\gamma} \leq n + 1.$$

We proceed further by considering the following partition of  $\mathcal{F}$  into subsets  $\mathcal{F}_i$ , each of them containing  $n$  functions:

$$(2.9) \quad \mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_l \cup \mathcal{F}_{\text{residual}},$$

where the partition is made in such a way that the set  $\mathcal{F}_{residual}$  contains only  $o(r^d)$  functions. This is possible to do because Theorem A tells us that  $\#\mathcal{F} = \frac{|T||\Omega|}{(2\pi)^d} r^d + o(r^d)$ . With each set  $\mathcal{F}_i$  associate  $h_i$  such that  $h_i \in \text{Ker}(P_{rT,\Omega})$  and such that

$$(2.10) \quad \langle h_i, h_j \rangle = \delta_{i,j}.$$

This can be done since  $\text{Ker}(P_{rT,\Omega})$  has infinite dimension, due to the inclusion  $\mathcal{D}(\mathbb{R}^d - rT) \subset \text{Ker}(P_{rT,\Omega})$ . Now, for each  $i$ , let  $\{\psi_j^{(i)}\}_{j=1}^{n+1}$  be a set of linear combinations of functions of  $\mathcal{F}$  such that

$$(2.11) \quad \langle \psi_k^{(i)}, \psi_j^{(i)} \rangle = \begin{cases} -\frac{1}{n+1} & \text{if } k \neq j \\ 1 - \frac{1}{n+1} & \text{if } k = j \end{cases},$$

which can be constructed using a linear algebra argument as in the next paragraph.

Consider a linear transformation  $U : \mathbb{R}^n \rightarrow \mathcal{F}_i$  mapping each vector of the canonical basis of  $\mathbb{R}^n$  to each of the given  $n$  orthogonal functions of  $\mathcal{F}_i$ . Let  $V$  be the subspace of  $\mathbb{R}^{n+1}$  which is orthogonal to the vector  $v_0 = \left[ \sqrt{\frac{1}{n+1}}, \dots, \sqrt{\frac{1}{n+1}} \right]^T \in \mathbb{R}^{n+1}$  and let  $\{v_1, \dots, v_n\}$  be an orthonormal basis of  $V$ . Clearly,  $\|v_0\| = 1$  and, for  $i = 1, \dots, n$ ,  $\langle v_0, v_i \rangle = 0$ . Thus, the matrix

$$Q = \begin{bmatrix} v_0 & v_1 & \dots & v_{n+1} \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}$$

is orthogonal. If  $u_1, \dots, u_{n+1} \in \mathbb{R}^{n+1}$  are the rows of  $Q$  then

$$Q^T = \begin{bmatrix} u_1 & \dots & u_{n+1} \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}$$

is also orthogonal, we have  $\langle u_i, u_j \rangle = \delta_{i,j}$ . Let  $u'_1, \dots, u'_{n+1} \in \mathbb{R}^n$  be the rows of  $Q$  without the elements of the first column. They satisfy

$$\langle u'_k, u'_j \rangle = \langle u_k, u_j \rangle - \frac{1}{n+1} = \begin{cases} -\frac{1}{n+1} & \text{if } k \neq j \\ 1 - \frac{1}{n+1} & \text{if } k = j \end{cases},$$

and the functions in (2.11) are obtained setting  $\psi_j^{(i)} = U u'_j$ .

We are now in a position to construct the desired orthonormal system. Define a sequence of orthonormal functions  $\{\Phi_j^{(i)}\}_{i=1}^l$  using the functions  $\psi_j^{(i)}$  from (2.11):

$$(2.12) \quad \Phi_j^{(i)} = \psi_j^{(i)} + \sqrt{\frac{1}{n+1}} h_i.$$

Since  $\psi_j^{(i)}$  are linear combinations of elements of  $\mathcal{F} = \{\phi_k\}$ , (2.12) is a representation of the form (2.3). Thus, (2.11) and (2.10) show that indeed  $\langle \Phi_k^{(i)}, \Phi_j^{(i)} \rangle = \delta_{k,j}$  and we can apply

(2.5), (2.8) and (2.6) to obtain

$$\begin{aligned}
\left\| P_{rT,\Omega} \Phi_j^{(i)} - \Phi_j^{(i)} \right\|^2 &\leq \sigma^2 \left\| \Phi_j^{(i)} \right\|^2 + (1 - \sigma^2) \left\| \sqrt{\frac{1}{n+1}} h_i \right\|^2 \\
&\leq \sigma^2 + (1 - \sigma^2) \gamma \\
&= \varepsilon.
\end{aligned}$$

Thus, the functions in  $\{\Phi_j^{(i)}\}_{j=1}^{n+1}$  verify (2.1) and  $\#\{\Phi_j^{(i)}\}_{j=1}^{n+1} = n+1$ . We have also  $\#\mathcal{F}_i = n$ , thus,

$$\#\{\Phi_j^{(i)}\}_{j=1}^{n+1} = \frac{n+1}{n} \#\mathcal{F}_i.$$

Now, the cardinality of the union of all the sequences  $\{\Phi_j^{(i)}\}$  obtained according to the above procedure is

$$\begin{aligned}
\# \left[ \bigcup_{i=1}^l \{\Phi_j^{(i)}\}_{j=1}^{n+1} \right] &= \frac{n+1}{n} \# \left[ \bigcup_{i=1}^l \mathcal{F}_i \right] \\
&= \frac{n+1}{n} \# [\mathcal{F} - \mathcal{F}_{residual}] \\
&= \frac{n+1}{n} (r^d (2\pi)^{-d} |T| |\Omega| + o(r^d)) \\
&\geq \frac{\frac{1}{\gamma} + 1}{\frac{1}{\gamma}} r^d (2\pi)^{-d} |T| |\Omega| + o(r^d) \\
&= (1 + \gamma) r^d (2\pi)^{-d} |T| |\Omega| + o(r^d).
\end{aligned}$$

We have used Proposition 1 in the third equality (the fact that the dimension of  $\mathcal{F}$  is  $r^d (2\pi)^{-d} |T| |\Omega| + o(r^d)$  and the fact that  $\mathcal{F}_{residual}$  contains only  $o(r^d)$  functions). Denote by  $M(rT, \Omega, \epsilon)$  the minimum number of orthonormal functions satisfying (2.1). By construction we have obtained

$$M(rT, \Omega, \epsilon) \geq \# \left[ \bigcup_{i=1}^l \{\Phi_j^{(i)}\}_{j=1}^{n+1} \right] \geq (1 + \gamma) r^d (2\pi)^{-d} |T| |\Omega| + o(r^d).$$

and now we take  $\sigma \rightarrow 0$ , so that  $\gamma \rightarrow \epsilon$  and we obtain

$$M(rT, \Omega, \epsilon) \geq \# \left[ \bigcup_{i=1}^l \{\Phi_j^{(i)}\}_{j=1}^{n+1} \right] \geq (1 + \epsilon) r^d (2\pi)^{-d} |T| |\Omega| + o(r^d).$$

This proves the lower inequality in (2.2).

Let us now prove the upper inequality in (2.2). Consider again  $f = \sum a_k \phi_k + h$  with  $h \in \text{Ker}(P_{rT,\Omega})$ . Then, using (2.4) and

$$\|B_\Omega D_{rT} f\|^2 = \langle P_{rT,\Omega} f, f \rangle = \sum |a_k|^2 \lambda_k,$$

together with the fact that  $D_{rT}$  is a projection, one can write

$$(2.13) \quad \|B_\Omega D_{rT} f - P_{rT,\Omega} f\|^2 = \|B_\Omega D_{rT} f\|^2 - \|P_{rT,\Omega} f\|^2 = \sum |a_k|^2 \lambda_k (1 - \lambda_k).$$



Now, for  $\delta > 0$  define  $\mathcal{E}(\delta)$  as the subspace generated by the eigenfunctions of  $P_{rT,\Omega}$  such that the corresponding eigenvalues satisfy  $\delta < \lambda_k < 1 - \delta$  and let

$$\mathcal{F}(\delta) = \left\{ f \in L^2(\mathbb{R}^d) : \|f\| = 1 \quad \sum_{\delta < \lambda_k < 1-\delta} |a_k|^2 \leq \delta \right\}.$$

For  $f \in \mathcal{F}(\delta)$ ,

$$\begin{aligned} & \|B_\Omega D_{rT} f - P_{rT,\Omega} f\|^2 \\ &= \sum_{\lambda_k \leq \delta} |a_k|^2 \lambda_k (1 - \lambda_k) + \sum_{\delta < \lambda_k < 1-\delta} |a_k|^2 \lambda_k (1 - \lambda_k) + \sum_{\lambda_k \geq 1-\delta} |a_k|^2 \lambda_k (1 - \lambda_k) \leq 3\delta. \end{aligned}$$

Thus,  $\delta$  can be chosen in such a way that

$$(2.14) \quad \|B_\Omega D_{rT} f - P_{rT,\Omega} f\|^2 \leq \varepsilon.$$

Let us assume the existence of a set  $\mathcal{N}$  of  $\eta(\epsilon, rT, \Omega)$  orthonormal functions of  $L^2(\mathbb{R}^d)$  satisfying (2.1). To estimate how many of them belong to  $\mathcal{F}(\delta)$ , consider two subspaces  $\mathcal{E}$  and  $\mathcal{G}$  with corresponding projections  $E, G$ , and dimensions  $e$  and  $g$  respectively, with  $e < g$ . Let  $v_1, \dots, v_g$  be an orthonormal set in  $\mathcal{G}$ . Then  $\sum \|Ev_i\|^2 = \sum (Ev_i, v_i) = \sum (GEGv_i, v_i)$  represents the trace of the operator  $GEG$ , independent of the choice of basis. Choose the basis  $\{w_i\}$  such that the first vectors are in  $\mathcal{GE}$  and the remaining vectors in the orthogonal complement in  $\mathcal{G}$  of  $\mathcal{GE}$  (the image of  $GE$ ). For each of the latter,  $(GEGw, w) = 0$ , while the dimension of  $\mathcal{GE}$  is at most  $e$ . Hence  $\sum \|Ev_i\|^2 = \sum_1^g (Ew_i, w_i) \leq \sum_1^e (GEGw_i, w_i) \leq e$ . Thus, the number of orthonormal vectors  $\{v_i\}$  for which  $\|Ev_i\|^2 \geq \delta$  cannot exceed  $e/\delta$ .

As a result of the previous paragraph, after excluding from  $\mathcal{N}$  at most  $\delta^{-1} \dim \mathcal{E}(\delta)$  elements, those remaining are in  $\mathcal{F}(\delta)$ . Since, from Theorem A, we have  $\dim \mathcal{E}(\delta) = o(r^d)$ , there are  $\eta(\epsilon, rT, \Omega) - o(r^d)$  functions in  $\mathcal{N} \cap \mathcal{F}(\delta)$ . Let  $f$  be one of them. Now we can use (2.1), (2.14) and the triangle inequality to obtain

$$1 - \|B_\Omega D_{rT} f\|^2 \leq \|B_\Omega D_{rT} f - f\| \leq 2\varepsilon,$$

leading to  $\|B_\Omega D_{rT} f\|^2 \geq 1 - 2\varepsilon$ , for each of the  $\eta(\epsilon, rT, \Omega) - o(r^d)$  orthonormal functions. Since  $\|B_\Omega D_{rT} f\|^2 = \langle P_{rT,\Omega} f, f \rangle$ , the sum of these terms for any orthonormal set cannot exceed the trace of  $D_{rT} B_\Omega D_{rT}$ . Thus, using the trace from Theorem A, we conclude that

$$(1 - 2\varepsilon) (\eta(\epsilon, rT, \Omega) - o(r^d)) \leq \sum_{k=0}^{\infty} \lambda_k(r, T, \Omega) = r^d (2\pi)^{-d} |T| |\Omega|,$$

leading to the upper inequality in (2.2). □

**Remark 1.** In the case where  $T$  and  $\Omega$  are finite unions of bounded intervals, the term  $o(r)$  in Theorem A can be replaced by  $\log r$  [15], [13]. Thus, (1.4) follows using this estimate in our proofs of Theorem 1. See the recent monograph [12] for more estimates on the eigenvalues of the time- and band- limiting operator.

**Remark 2.** *It is possible to obtain an analogue of Theorem 1 in the set up of the Hankel transform. The result corresponding to Theorem A has been proved in [1].*

**Remark 3.** *The proof of the lower inequality in (2.2) constructs a new set of orthogonal functions. On the one side we don't know yet to what extent such functions can be used in applications. On the other side the lower inequality in (2.2) may provide useful information in cases where signals are approximated by functions which are not optimal concentrated as the prolates, but still have some concentration properties. This is the case of the Hermite functions, where an estimate of the energy left outside  $\Omega$  may provide an indication of the increase in the number of functions required to avoid undersampling.*

### 3. GABOR LOCALIZATION OPERATORS

The Gabor (or short-time Fourier) transform of a function or distribution  $f$  with respect to a window function  $g \in L^2(\mathbb{R}^d)$  is defined to be, for  $z = (x, \xi) \in \mathbb{R}^{2d}$ :

$$(3.1) \quad \mathcal{V}_g f(z) = \mathcal{V}_g f(x, \xi) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \xi t} dt.$$

The following relations are usually called the orthogonal relations for the short-time Fourier transform. Let  $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$ . Then  $V_{g_1} f_1, V_{g_2} f_2 \in L^2(\mathbb{R}^{2d})$  and

$$(3.2) \quad \int \int_{\mathbb{R}^{2d}} V_{g_1} f_1(x, \xi) \overline{V_{g_2} f_2(x, \xi)} dx d\xi = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2 \rangle_{L^2(\mathbb{R}^d)}}.$$

The localization operator which concentrates the time-frequency content of a function in the region  $S$  operator  $\mathcal{C}_S : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  can be defined weakly as

$$\langle \mathcal{C}_S f, h \rangle = \int \int_S \mathcal{V}_g f(x, \xi) \overline{\mathcal{V}_g h(x, \xi)} dx d\xi,$$

for all  $f, g \in L^2(\mathbb{R}^d)$ . These operators have been introduced in time-frequency analysis by Daubechies [4]. Since then, applications and connections to several mathematical topics, namely complex and harmonic analysis [18], [2], [3], [9] have been found. The eigenvalue problem has been object of a detailed study in [17], [8] and [6].

The image of  $L^2(\mathbb{R}^d)$  under the Gabor transform with the window  $g$  will be named as the *Gabor space*  $\mathcal{G}_g$ . It is the following subspace of  $L^2(\mathbb{R}^{2d})$ :

$$\mathcal{G}_g = \{V_g f : f \in L^2(\mathbb{R}^d)\}.$$

The reproducing kernel of the Gabor space  $\mathcal{G}_g$  is

$$(3.3) \quad K_g(z, w) = \langle \pi_z g, \pi_w g \rangle_{L^2(\mathbb{R}^d)}$$

and the projection operator  $\mathcal{P}_g : L^2(\mathbb{R}^{2d}) \rightarrow \mathcal{G}_g$ ,

$$\mathcal{P}_g F(z) = \int F(w) \overline{K_g(z, w)} dw.$$

It is shown in [17] that, for  $F \in \mathcal{G}_g$ ,

$$\mathcal{V}_g \mathcal{C}_S \mathcal{V}_g^{-1} F(z) = \int_S F(w) \overline{K_g(z, w)} dw = \mathcal{P}_g D_S F(z).$$

For the whole  $L^2(\mathbb{R}^{2d})$  one can write

$$\mathcal{V}_g \mathcal{C}_S \mathcal{V}_g^* = \mathcal{P}_g D_S.$$

Thus, the spectral properties of  $\mathcal{C}_S$  are identical to those of  $\mathcal{P}_g D_S$ . Moreover, the operator  $D_S \mathcal{P}_g D_S$  in  $L^2(\mathbb{R}^{2d})$  and the operator  $\mathcal{P}_g D_S$  have the same nonzero eigenvalues with multiplicity (see Lemma 1 in [17]). The analogue of Theorem A in this context is the following.

**Theorem B** [17]. *The operator  $D_{rS} \mathcal{P}_g D_{rS}$  is bounded by 1, self-adjoint, positive, and completely continuous. Denoting its set of eigenvalues, arranged in nonincreasing order, by  $\{\lambda_k(rS)\}$ , we have*

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda_k(rS) &= r^d |S| \\ \sum_{k=0}^{\infty} \lambda_k^2(rS) &= r^d |S| - o(r^d). \end{aligned}$$

Moreover, given  $0 < \gamma < 1$ , the number  $M_r(\gamma)$  of eigenvalues which are not smaller than  $\gamma$ , satisfies, as  $r \rightarrow \infty$ ,

$$M_r(\gamma) = r^d |S| + o(r^d).$$

Now that we have described the Gabor set-up in a close analogy to the band- time- limiting case, we obtain an analogue of Theorem 1 by performing minor adaptations in the proof.

**Theorem 2.** *Let  $\eta(\epsilon, rS)$  stand for the maximum number of orthogonal functions  $F \in L^2(\mathbb{R}^{2d})$  such that*

$$(3.4) \quad \|D_{rS} \mathcal{P}_g D_{rS} F - F\|^2 \leq \epsilon.$$

*Then, as  $r \rightarrow \infty$ , the following inequalities hold:*

$$|S| (1 + \epsilon) \leq \lim_{r \rightarrow \infty} \frac{\eta(\epsilon, rS)}{r^{2d}} \leq \frac{|S|}{1 - 2\epsilon}.$$

*Proof.* The proof mimics the proof of Theorem 1, replacing  $D_{rT} B_{\Omega} D_{rT}$  by  $D_{rS} \mathcal{P}_g D_{rS}$ ,  $B_{\Omega} D_{rT}$  by  $\mathcal{P}_g D_{rS}$  and Theorem A by Theorem B.  $\square$

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